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A Second-Order Singular Boundary Value Problem

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Abstract—We study the second-order boundary value problem

$$-u''(t) = a(t)f(u(t)), \qquad 0 < t < 1,$$

satisfying

$$\alpha u(0) - \beta u'(0) = 0,$$
$$\gamma u(1) + \delta u'(1) = 0,$$

where $a(t) = \prod_{i=1}^{n} a_i(t)$ and $\alpha, \beta, \gamma, \delta \geq 0$, $\alpha\gamma + \alpha\delta + \beta\gamma > 0$. We assume that each $a_i(t) \in L^{p_i}[0, 1]$ for $p_i \geq 1$ and that each $a_i(t)$ has a singularity in $(0, 1)$. To show the existence of countably many positive solutions, we apply Hölder's inequality and Krasnosel'skiĭ's fixed-point theorem for operators on a cone.

Keywords—Boundary value problem, Fixed-point theorem, Green's function, Hölder's inequality, Multiple solutions.

1. INTRODUCTION

We consider the existence of infinitely many positive solutions for a second-order singular boundary value problem

$$-u''(t) = a(t)f(u(t)), \qquad 0 < t < 1, \tag{1}$$

$$\alpha u(0) - \beta u'(0) = 0, \tag{2}$$

$$\gamma u(1) + \delta u'(1) = 0, \tag{3}$$

where $a(t) = \prod_{i=1}^{n} a_i(t)$ and $\alpha, \beta, \gamma, \delta \geq 0$, $\alpha\gamma + \alpha\delta + \gamma\delta > 0$.

This paper is organized in the following fashion. Our approach is based on the properties of the Green's function associated with (1)–(3). In Section 2, we give the required properties of the Green's function and state the theorems that will be used to establish our main results presented

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in Section 3. In Section 4, we provide examples of boundary value problems illustrating our main results with the functions of the form

$$a(t) = \prod_{i=1}^{n} a_i(t), \qquad a_i \in L^{p_i}[0,1], \quad p_i \geq 1,$$

for which

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1 \qquad \text{and} \qquad \sum_{i=1}^{n} \frac{1}{p_i} > 1.$$

Due to their origins and applications, nonlinear boundary value problems admitting positive solutions have received a great deal of attention. Second-order problems are induced and motivated by nonlinear elliptic problems in annular domains. In [1], Bohner *et al.* gave a nice overview of their origins and applications to biology, chemistry, and physics. Second-order problems also include p-Laplacian semipositone problems for which topological degree methods serve as one of the tools of analysis [2]. The authors of the above-mentioned paper have obtained several exact multiplicity results. Topological degree approach have led to several theorems based on cone-theoretic methods on Banach spaces. Specifically, there have been recently obtained numerous results based on fixed-point theorems due to Krasnosel'skiĭ [3] (also see [4]), Leggett-Williams [5], and their extensions (e.g., [6,7]). Other investigations were done using such techniques as, for example, shooting methods and methods of numerical analysis (see [8]). The existence results concerned various types of boundary value problems for both differential [9–14] and difference equations [15–17], of second and higher order. Earlier results [9,11,12] hinged on applications of Krasnosel'skiĭ's fixed-point theorem and dealt with boundary value problems which guaranteed the existence of at least one solution. In [1,18], the authors considered nonlinear boundary value problems, where the inhomogeneous term was either sublinear or superlinear, that is when for

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u},$$

we have $f_0 = \infty$, $f_\infty = 0$ or $f_0 = 0$, $f_\infty = \infty$, respectively.

Subsequently, this approach allowed the possibility of constructing growth conditions on $f(t,u)$ under which the existence of arbitrary finitely many solutions was established (see [10]). In [1], Bohner *et al.* analyzed $(k, n-k)$ conjugate problems. Their approach was based on thorough analysis of the corresponding Green's function and their results extended the works by Eloe and Henderson [11]. The theorems due to Leggett-Williams [5] and Avery [6] were used to obtain triple positive solutions. Such constructions were made by Davis and Henderson [19] and Avery [6].

The continuity of the inhomogeneous term served as a standing assumption for many papers. Along with these, singular boundary value problems have been investigated [8,12,16,17,20–22]. For many other results on nonlinear singular problems for both differential and difference equations, we direct the reader to thoroughly written books by Agarwal *et al.* [23] and O'Regan [24].

The study has recently evolved to determining the existence of infinitely many solutions of boundary value problems [21,22,25,26]. Eloe, Henderson and Kosmatov [25], established the existence of infinitely many solutions for $(k, n-k)$ conjugate type problem. In [21,22], singular second-order problems were studied. These results were extended in [26] to the case when the inhomogeneous term had infinitely many singularities in $(0,1)$.

In Kaufmann and Kosmatov [26], the authors showed the existence of multiple positive solutions of (1) when $a(t) \in L^p[0,1]$ for some $p \geq 1$. In particular, they considered the case where $a(t)$ is an infinite series of singular functions. Our approach is different from the traditional way of obtaining the estimates from "above" required to apply Hölder's inequality. When considering the inhomogeneous singular term in the form $a(t)f(u(t))$, the integrability of $a(t)$ is merely assumed. Then, the common C^∞-norm estimates on the Green's function were done. This becomes impossible if $a(t)$ is in L^p, $p > 1$. Based on Hölder's inequality, our approach enables us study more general classes of singular problems.

We assume that f is a nonnegative continuous function and for each $i = 1, 2, \ldots, n$, $a_i(t)$ is nonnegative and satisfies the following conditions:

(A1) $\lim_{t \to t_i} a_i(t) = \infty$, where $0 < t_n < t_{n-1} < \cdots < t_1 < 1$;
(A2) there exists $m_i > 0$ such that $a_i(t) \geq m_i$ a.e. on $[0, 1]$;
(A3) there exists a $p_i \geq 1$ such that $a_i(t) \in L^{p_i}[0, 1]$.

We show that if $\sum_{i=1}^{n} 1/p_i \leq 1$, $\{a_i(t)\}$ satisfies Conditions (A1)–(A3) and f satisfies oscillatory like growth conditions about a wedge, then the boundary value problem (1)–(3) has infinitely many solutions.

If $\sum_{i=1}^{n}(1/p_i) > 1$, then, in addition to (A1)–(A3), we need to impose the following condition on the functions $a_i(t)$:

(A4) for each $i = 1, 2, \ldots, n, a_i(t)$ is continuous a.e. on $[0, 1]$.

2. PRELIMINARIES

It is well known that fixed points of the operator

$$Tu(t) = \int_0^1 G(t, s) \prod_{i=1}^{n} a_i(s) f(u(s))\, ds, \qquad 0 \leq t \leq 1, \tag{4}$$

are solutions of (1)–(3), where $G(t, s)$ denotes the Green's function of

$$-u'' = 0$$

satisfying (2) and (3). The Green's function is

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\alpha t + \beta)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1, \\ (\alpha s + \beta)(\gamma + \delta - \gamma t), & 0 \leq s \leq t \leq 1, \end{cases} \tag{5}$$

where $\rho = \alpha\gamma + \alpha\delta + \beta\gamma$.

We need some properties of (5) in order to establish the existence of our fixed points. For $0 < \tau < 1/2$, define

$$L_\tau = \min\left\{\frac{\alpha\tau + \beta}{\alpha + \beta}, \frac{\gamma\tau + \delta}{\gamma + \delta}\right\}.$$

It can be easily shown that

$$\min_{t \in [\tau, 1-\tau]} G(t, s) \geq L_\tau G(t', s), \tag{6}$$

for all $t', s \in [0, 1]$.

By $L^p[a, b]$, we denote the space of Lebesgue integrable functions with the norm, $\|\cdot\|_p$, defined for $1 \leq p < \infty$ and $p = \infty$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p\, dx\right)^{1/p}$$

and

$$\|f\|_\infty = \operatorname{ess.sup} |f(t)|,$$

respectively.

Define

$$\Gamma_1 = \begin{cases} \left(1+\dfrac{\delta}{\gamma}\right)^2, & \alpha = 0,\ \beta\gamma > 0, \\ \left(1+\dfrac{\beta}{\alpha}\right)^2, & \gamma = 0,\ \alpha\delta > 0, \\ \left(1+\dfrac{\beta}{\alpha}+\dfrac{\delta}{\gamma}\right)^2, & \alpha\gamma > 0 \end{cases}$$

and

$$\Gamma_2 = \begin{cases} 1+\dfrac{\delta}{\gamma}, & \alpha = 0,\ \beta\gamma > 0, \\ 1+\dfrac{\beta}{\alpha}, & \gamma = 0,\ \alpha\delta > 0, \\ \dfrac{1}{4}\left(1+\dfrac{\beta}{\alpha}+\dfrac{\delta}{\gamma}\right), & \alpha\gamma > 0 \text{ and } 0 \le \dfrac{\alpha\gamma+\alpha\delta-\gamma\beta}{2\alpha\gamma} \le 1, \\ \dfrac{\beta(\gamma+\delta)}{\rho}, & \alpha\gamma > 0 \text{ and } \dfrac{\alpha\gamma+\alpha\delta-\gamma\beta}{2\alpha\gamma} < 0, \\ \dfrac{(\alpha+\beta)\delta}{\rho}, & \alpha\gamma > 0 \text{ and } \dfrac{\alpha\gamma+\alpha\delta-\gamma\beta}{2\alpha\gamma} > 1. \end{cases}$$

The proof of the following technical lemma is given in [22].

LEMMA 1. *The Green's function $G(t,s)$ satisfies*

$$\max_{t\in[0,1]} \int_\tau^{1-\tau} G(t,s)\,ds \ge \frac{1}{4}\left(\frac{1}{4}-\tau\right). \tag{7}$$

Furthermore, for all $q > 0$, $G(t,\cdot) \in L^q[0,1]$ and

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_q \le \Gamma_1. \tag{8}$$

In addition,

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_\infty = \Gamma_2. \tag{9}$$

To establish the existence of fixed points of (4), we employ Krasnosel'skiĭ's fixed-point theorem.

DEFINITION 1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be closed and nonempty. Then, $\mathcal{P}$ is said to be a cone if*

(1) *$\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and for all $\alpha, \beta \ge 0$, and*
(2) *$u, -u \in \mathcal{P}$ implies $u = 0$.*

THEOREM 2. KRASNOSEL'SKIĬ'S. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) *$\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, or*
(ii) *$\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.*

Then, T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We use (6) to define our cones. For our Banach space, let $\mathcal{B} = C[0,1]$ be endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. For $\tau \in [0, 1/2)$, define the cone $\mathcal{P}_\tau \subset \mathcal{B}$ by

$$\mathcal{P}_\tau = \left\{ u(t) \in \mathcal{B} \mid u(t) \geq 0, \text{ on } [0,1], \min_{t\in[\tau,1-\tau]} u(t) \geq L_\tau \|u\| \right\}. \tag{10}$$

Clearly, (4) is continuous and compact and as such is completely continuous. Also, it is easily shown that (4) is cone preserving.

In order to establish some of the norm inequalities in Theorem 2, we need Hölder's inequality.

THEOREM 3. HÖLDER. *Let $f_i \in L^{p_i}[a,b]$ with $p_i > 1$, for $i = 1, 2, \dots, n$, and $\sum_{i=1}^n (1/p_i) = 1$. Then, $\prod_{i=1}^n f_i \in L^1[a,b]$ and*

$$\left\| \prod_{i=1}^n f_i \right\|_1 \leq \prod_{i=1}^n \|f_i\|_{p_i}.$$

Let $f \in L^1[a,b]$ and $g \in L^\infty[a,b]$. Then, $fg \in L^1[a,b]$ and

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

3. POSITIVE SOLUTIONS

First, we consider the case $\sum_{i=1}^n (1/p_i) < 1$.

THEOREM 4. *Assume that $a_i(t)$, $i = 1, 2, \dots, n$, satisfy (A1)–(A3). Let $\{\tau_k\}_{k=1}^\infty$ be such that $0 < \tau_1 < 1/2$, $\tau_k \downarrow \tau^*$ and $0 < \tau^* < t_n$. Let $\{A_k\}_{k=1}^\infty$ and $\{B_k\}_{k=1}^\infty$ be such that*

$$A_{k+1} < L_{\tau_k} B_k < B_k < C B_k < A_k, \qquad k \in \mathbb{N},$$

where

$$C = \max \left\{ \frac{8}{(1 - 2\tau_1) \prod_{i=1}^n m_i}, 1 \right\}. \tag{11}$$

Assume that f satisfies

(H1a) *$f(z) \leq M_1 A_k$ for all $z \in [0, A_k]$, $k \in \mathbb{N}$, where $M_1 \leq 1/(\Gamma_1 \prod_{i=1}^n \|a_i\|_{p_i})$.*

(H2) *$f(z) \geq C B_k$ for all $z \in [L_{\tau_k} B_k, B_k]$.*

Then, the boundary value problem (1)–(3) has infinitely many solutions $\{u_k\}_{k=1}^\infty$. Furthermore, $B_k \leq \|u_k\| \leq A_k$ for each $k \in \mathbb{N}$.

PROOF. For a fixed k, define $\Omega_{1,k} = \{u \in \mathcal{B} : \|u\| < A_k\}$ and let $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{1,k}$, where $\mathcal{P}_{\tau_k}$ is given by (10) with $\tau = \tau_k$. Then,

$$u(s) \leq A_k = \|u\|,$$

for all $s \in [0,1]$. By (H1a),

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) a(s) f(u(s))\, ds \\ &\leq \max_{t\in[0,1]} \int_0^1 G(t,s) \prod_{i=1}^n a_i(s)\, ds\, M A_k. \end{aligned}$$

There exists a $q > 1$ such that $1/q + \sum_{i=1}^n (1/p_i) = 1$. By the first part of Theorem 3,

$$\|Tu\| \leq \max_{t\in[0,1]} \|G(t,\cdot)\|_q \prod_{i=1}^n \|a\|_{p_i} M_1 A_k.$$

From (8) and (H1a),

$$\begin{aligned}\|Tu\| &\leq \Gamma_1 \prod_{i=1}^{n} \|a\|_{p_i} M_1 A_k \\ &\leq A_k.\end{aligned}$$

Since $\|u\| = A_k$ for all $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{1,k}$, then

$$\|Tu\| \leq \|u\|. \tag{12}$$

REMARK. Since $1/2 - t_1 < 1 < \Gamma_1$ and $\prod_{i=1}^{n} m_i \leq \prod_{i=1}^{n} \|a_i\|_{p_i}$, then

$$M_1 \leq \frac{1}{\Gamma_1 \prod_{i=1}^{n} \|a_i\|_{p_i}} < \frac{8}{(1-2\tau_1)\prod_{i=1}^{n} m_i} \leq C$$

(otherwise the theorem is vacuously true).

Now define $\Omega_{2,k} = \{u \in \mathcal{B} : \|u\| < B_k\}$. Let $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{2,k}$ and let $s \in [\tau_k, 1-\tau_k]$. Then, by (6),

$$B_k = \|u\| \geq u(s) \geq \min_{[\tau_k, 1-\tau_k]} u(s) \geq L_{\tau_k}\|u\| = L_{\tau_k} B_k.$$

By (H2),

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)f(u(s))\,ds \\ &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)\,ds\,CB_k.\end{aligned}$$

Observe that since $\tau^* < t_n$, then at least one t_i is contained in each interval $[\tau_k, 1-\tau_k]$ for k large enough. Now, by (A2) and (7),

$$\begin{aligned}\|Tu\| &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)\,ds\,CB_k \\ &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)\,ds \prod_{i=1}^{n} m_i CB_k \\ &= \frac{1}{4}\left(\frac{1}{2} - \tau_k\right) \prod_{i=1}^{n} m_i CB_k.\end{aligned}$$

Recall that $C \geq 8/((1-2\tau_1)\prod_{i=1}^{n} m_i)$ and $\tau_k < \tau_1$. Thus, if $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{2,k}$, then

$$\|Tu\| \geq \frac{1}{4}\left(\frac{1}{2} - \tau_k\right) \prod_{i=1}^{n} m_i CB_k \geq B_k = \|u\|. \tag{13}$$

Now $0 \in \Omega_{2,k} \subset \bar{\Omega}_{2,k} \subset \Omega_{1,k}$. By (11),(12), it follows from Theorem 2 that the operator T has a fixed point $u_k \in \mathcal{P}_{\tau_k} \cap (\bar{\Omega}_{1,k} \setminus \Omega_{2,k})$ such that $B_k \leq \|u_k\| \leq A_k$. Since $k \in \mathbb{N}$ was arbitrary, the proof is complete. ∎

Note that the growth constant C given by (12) is also involved in the assumptions of Theorems 5 and 7.

Now we deal with the case $\sum_{i=1}^{n}(1/p_i) = 1$.

THEOREM 5. *Assume that* $a_i(t)$, $i = 1, 2, \dots, n$, *satisfy (A1)–(A3). Let* $\{\tau_k\}_{k=1}^{\infty}$ *be such that* $0 < \tau_1 < 1/2$, $\tau_k \downarrow \tau^* > 0$, *and* $\tau^* < t_n$. *Let* $\{A_k\}_{k=1}^{\infty}$ *and* $\{B_k\}_{k=1}^{\infty}$ *be such that*

$$A_{k+1} < L_{\tau_k} B_k < B_k < C B_k < A_k, \qquad k \in \mathbb{N}.$$

Assume that f satisfies

(H1b) $f(z) \le M_2 A_k$ *for all* $z \in [0, A_k]$, $k \in \mathbb{N}$, *where* $M_2 < \min\{1/(\Gamma_2 \prod_{i=1}^{n} \|a_i\|_{p_i}), C\}$

and (H2). Then, the boundary value problem (1)–(3) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$. *Furthermore,* $B_k \le \|u_k\| \le A_k$ *for each* $k \in \mathbb{N}$.

PROOF. For a fixed k, let $\Omega_{1,k}$ be as in the proof of Theorem 4 and let $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{1,k}$. Again

$$u(s) \le A_k = \|u\|,$$

for all $s \in [0,1]$. By (H1b), Theorem 4 and (9),

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) a(s) f(u(s))\, ds \\
&\le \max_{t\in[0,1]} \int_0^1 G(t,s) \prod_{i=1}^{n} a_i(s)\, ds\, M_2 A_k \\
&\le \max_{t\in[0,1]} \|G(t,\cdot)\|_\infty \prod_{i=1}^{n} \|a_i\|_{p_i} M_2 A_k \\
&< A_k \\
&= \|u\|.
\end{aligned}$$

Thus,

$$\|Tu\| < \|u\|,$$

for $u \in \mathcal{P}_k \cap \partial\Omega_{1,k}$.

Now define $\Omega_{2,k} = \{u \in \mathcal{B} : \|u\| < B_k\}$. Let $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{2,k}$ and let $s \in [\tau_k, 1-\tau_k]$. Then, the argument leading to (12) carries over to the present case and completes the proof. ∎

Before we prove our last result, we need a lemma establishing the convergence of the integral $\int_0^1 \prod_{i=1}^{n} a_i(s)\, ds$ when $\sum_{i=1}^{n}(1/p_i) > 1$.

LEMMA 6. *Assume that* $a_i(t)$ *satisfies (A1), (A3), and (A4). Let* $\sum_{i=1}^{n}(1/p_i) > 1$. *Then,*

$$0 < \int_0^1 \prod_{i=1}^{n} a_i(s)\, ds < \infty.$$

PROOF. For each $i = 1, \dots, n$, let E_i be such that $t_i \in E_i$, $t_j \in [0,1] \setminus \bar{E}_i$ if $i \ne j$, E_i has positive measure and $\cup \bar{E}_i = [0,1]$. Then,

$$\int_0^1 \prod_{j=1}^{n} a_j(s)\, ds \le \sum_{i=1}^{n} \int_{E_i} \prod_{j=1}^{n} a_j(s)\, ds.$$

Fix i. Then,

$$\int_{E_i} \prod_{j=1}^{n} a_j(s)\, ds = \int_{E_i} \left[\prod_{j=1, j\ne i}^{n} a_j(s) \right] a_i(s)\, ds.$$

Since $t_j \notin \tilde{E}_i$, then by (A4), $\| \prod_{j=1,j\neq i}^{n} a_j(s) \|_{E_i,\infty} \equiv M_i < \infty$. Hence,

$$\begin{aligned}\int_{E_i} \prod_{j=1}^{n} a_j(s)\,ds &\leq M_i \int_{E_i} a_i(s)\,ds \\ &< M_i \int_0^1 a_i(s)\,ds \\ &= M_i \|a_i\|_1.\end{aligned}$$

Thus,

$$\int_0^1 \prod_{j=1}^{n} a_j(s)\,ds \leq \sum_{i=1}^{n} M_i \|a_i\|_1 < \infty,$$

which completes the proof. ∎

Finally, let $\sum_{i=1}^{n}(1/p_i) > 1$.

THEOREM 7. *Assume that $a_i(t)$ satisfies Conditions (A1)–(A4). Let $\{\tau_k\}_{k=1}^{\infty}$ be such that $0 < \tau_1 < 1/2$, $\tau_k \downarrow \tau^* > 0$ and $\tau^* < t_n$. Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be such that*

$$A_{k+1} < L_{\tau_k} B_k < B_k < C B_k < A_k, \qquad k \in \mathbb{N}.$$

Assume that f satisfies
(H1c) *$f(z) \leq M_3 A_k$ for all $z \in [0, A_k]$, $k \in \mathbb{N}$, where $M_3 < \min\{1/(\Gamma_2 \int_0^1 \prod_{j=1}^{n} a_j(s)\,ds), C\}$ and (H2). Then, the boundary value problem (1)–(3) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$. Furthermore, $B_k \leq \|u_k\| \leq A_k$ for each $k \in \mathbb{N}$.*

PROOF. Note that $M_3 < \infty$. By (H1c),

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) \prod_{j=1}^{n} a_j(s) f(u(s))\,ds \\ &\leq \int_0^1 \prod_{j=1}^{n} a_j(s)\,ds \max_{t\in[0,1]} \|G(t,\cdot)\|_\infty M_3 A_k \\ &\leq \Gamma_2 \int_0^1 \prod_{j=1}^{n} a_j(s)\,ds M_3 A_k \\ &< A_k.\end{aligned}$$

This shows that if $u \in \mathcal{P}_k \cap \partial\Omega_{1,k}$, where $\Omega_{1,k} = \{u \in \mathcal{B} : \|u\| < A_k\}$, then

$$\|Tu\| \leq \|u\|.$$

Define $\Omega_{2,k} = \{u \in \mathcal{B} : \|u\| < B_k\}$ and let $u \in \mathcal{P}_k \cap \partial\Omega_{2,k}$. Then, the argument employed in the proof of Theorem 4 applies directly to yield

$$\|Tu\| \geq \|u\|.$$

By Theorem 2, the proof is now complete. ∎

REMARK. Observe that the conditions on sequences $\{A_k\}$ and $\{B_k\}$ in Theorem 4 imply that $A_k \downarrow 0$ and $B_k \downarrow 0$. Consequently, the solutions u_k of (1)–(3) satisfy $\|u_k\| \downarrow 0$.

4. EXAMPLES

In this section, we provide examples of families of functions $a_i(t)$ satisfying Conditions (A1)–(A4) corresponding to the cases $\sum_{i=1}^{n}(1/p_i) < 1$ and $\sum_{i=1}^{n}(1/p_i) > 1$. We take $\tau_k = 1/8 + 1/3k$, $k \in \mathbb{N}$ and consider the boundary value problem ($\alpha = \gamma = 1,\ \beta = \delta = 0$)

$$-u''(t) = a_1(t)a_2(t)f(u(t)), \qquad 0 < t < 1, \tag{14}$$

$$u(0) = u(1) = 0. \tag{15}$$

For this boundary value problem, we have $\Gamma_1 = 1$, $\Gamma_2 = 1/4$, and $L_\tau = \tau$. The function f is given by

$$f(u) = \begin{cases} M, & u > A_1, \\ CB_k + \dfrac{MA_k - CB_k}{A_k - B_k}(u - B_k), & B_k \le u \le A_k,\ k \in \mathbb{N}, \\ CB_k, & \tau_k B_k < u < B_k,\ k \in \mathbb{N}, \\ MA_{k+1} + \dfrac{CB_k - MA_{k+1}}{\tau_k B_k - A_{k+1}}(u - A_{k+1}), & A_{k+1} < u \le \tau_k B_k,\ k \in \mathbb{N}, \\ 0, & u = 0, \end{cases}$$

where $A_1 = 1$, $A_{k+1} = 12/(4 + (k+1)^2)! + (1/16 + 1/6k)(1/(4 + k^2)!)$ and $B_k = 1/(4 + k^2)!$, $k \in \mathbb{N}$. Note that

$$A_{k+1} < \tau_k B_k < B_k < CB_k < A_k, \qquad k \in \mathbb{N}.$$

EXAMPLE 1. For our first example, set $a_1(t) = 1/(|t - 1/4|^{1/4})$ and $a_2(t) = 1/(|t - 3/4|^{1/4})$. Then, $a_1, a_2 \in L^p[0,1]$ for all $0 < p < 4$. In particular, let $0 < \varepsilon < 1$ be fixed. Then, $a_1, a_2 \in L^{3+\varepsilon}[0,1]$. Observe that $2/(3+\varepsilon) < 1$ for all $0 < \varepsilon < 1$. Now, $\|a_1\|_{3+\varepsilon} = \|a_2\|_{3+\varepsilon} = \sqrt{2}[(1 + 3^{(1-\varepsilon)/4})/(1-\varepsilon)]^{1/(3+\varepsilon)}$. For these functions, $m_1 = m_2 = (4/3)^{1/4}$. If

$$M \le \frac{1}{\Gamma_1 \|a_1\|_{3+\varepsilon} \|a_2\|_{3+\varepsilon}} = \frac{1}{2}\left[\frac{1-\varepsilon}{1 + 3^{(1-\varepsilon)/4}}\right]^{2/(3+\varepsilon)}$$

and

$$C = \max\left\{\frac{8}{(1 - 2\tau_1)m_1 m_2}, 1\right\} = 48\sqrt{3}.$$

(Note that $0 < M < C$ if $0 < \varepsilon < 1$), then by Theorem 4 the boundary value problem (14),(15) has infinitely many solutions u_k such that $B_k \le \|u_k\| \le A_k$ for each $k \in \mathbb{N}$.

EXAMPLE 2. For our second example, set $a_1(t) = 1/(|t - 1/4|^{1/2})$ and $a_2(t) = 1/(|t - 3/4|^{1/2})$. Then, $a_1, a_2 \in L^p[0,1]$ for all $0 < p < 2$. In particular, if $0 < \varepsilon < 1$ is fixed, then $a_1, a_2 \in L^{1+\varepsilon}[0,1]$. Note that $2/(1+\varepsilon) > 1$ for all $0 < \varepsilon < 1$. A simple calculation shows that $\int_0^1 a_1(t)a_2(t)\,dt = \pi - \ln(7 - 4\sqrt{3})$. For these functions, $m_1 = m_2 = (4/3)^{1/2}$. If

$$M \le \pi - \ln\left(7 - 4\sqrt{3}\right)$$

and

$$C = \max\left\{\frac{8}{(1 - 2\tau_1)m_1 m_2}, 1\right\} = 72.$$

(Note that $M < C$.) Then by Theorem 7 the boundary value problem (14),(15) has infinitely many solutions u_k such that $B_k \le \|u_k\| \le A_k$ for each $k \in \mathbb{N}$.

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